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NOTATION

B = displacement of interface from its equilibrium position
g = gravitational acceleration
N_{BO} = Bond number = $\delta^2 g(\rho_L - \rho_v)/\sigma$
N_{CR} = Crispation number = $\mu_L \kappa_L/\sigma\delta$
N_H = Hickman number = $-(\partial\eta/\partial T)\eta\delta\Delta T\mu_v(\rho_v^{-1} - \rho_L^{-1})/\rho_L\kappa_L\sigma$
N_p = pressure number = $-(\partial\eta/\partial P_v)\eta\mu_L\mu_v(\rho_v^{-1} - \rho_L^{-1})/\rho_L\sigma\delta$
N_{PR} = Prandtl number = $\mu_L/\rho_L\kappa_L$
N_{RE} = Reynolds number = $\eta\delta/\mu_L$
N_ρ = density ratio = ρ_L/ρ_v
N_μ = viscosity ratio = μ_L/μ_v
P = pressure
T = temperature

Greek Letters

α = dimensionless wave number of the disturbance
δ = depth of the thermal boundary layer
η = mass flux of evaporation
κ = thermal diffusivity
μ = fluid viscosity
ρ = fluid density
σ = surface tension

Subscripts

L = liquid phase
v = vapor phase

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Stability of Liquid Films Method of Quadrature by Differentiation

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The flow of a thin liquid film down an inclined plane is frequently used as an idealized hydrodynamic description of the complex flow occurring in numerous industrial apparatuses (evaporators, condensers, rocket engines, nuclear reactors) and in some natural processes (soil erosion, transport of materials by rain water, spacecraft thermal shield melting). For a complete description, the stability of the interface has to be investigated because of the profound effect the waves may have on heat and mass transfer rates. The linear stability analysis of a thin liquid film flow by the perturbation method leads to an eigenvalue system. This system includes the well known Orr-Sommerfeld differential equation and four boundary conditions. Many attempts have been made to solve this system. The analytical studies of Yih (1955), Benjamin (1957), Yih (1963), Anshus and Goren (1966), Lin (1967), Krantz and Goren (1971), Krantz and Owens (1973), and Shuler and Krantz (1976) and the numerical calculations of Whitaker (1964), De Bruin (1974), and Sterling and Towell (1965) may be mentioned, in particular. However, we note that none of the analytical approaches offers a general solution; they are either limited in their field of application or based on approximations which are not always justified.

The purpose of the following is to find an analytical solution for the previous eigenvalue system which is valid for not only small values of the wave number ($\alpha < 0.3$) and which simply requires an analytic stream function on the interval [0;1]. This solution is based upon the method of quadrature by differentiation (Lanczos, 1956).

FORMULATION OF THE PROBLEM

The temporal formulation of the well-known dimensionless Orr-Sommerfeld equation is given by

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha Re\,[(\bar{u} - c)\,(\phi'' - \alpha^2\phi) - \bar{u}''\phi] \tag{1}$$

The boundary conditions as formulated by Benjamin (1957) are

$$\phi'(1) = 0 \tag{2}$$

$$\phi(1) = 0 \tag{3}$$

$$\phi''(0) + \left(\alpha^2 - \frac{3}{c'}\right)\phi(0) = 0 \tag{4}$$

$$\frac{\alpha(3\cot\theta + \alpha^2\, We\, Re)}{c'}\phi(0) + \alpha(Re\, c' + 3\alpha i)\,\phi'(0) - i\phi'''(0) = 0 \tag{5}$$

with

$$c' \triangleq c - \frac{3}{2};\quad \bar{u} = \frac{3}{2}(1 - y^2) \tag{6}$$

Equation (1) and the four boundary conditions (2) to (5) constitute an eigenvalue system which has a nontrivial solution if a relation exists between the five parameters θ, Re, We, α, c:

$$F(\theta, Re, We, \alpha, c) = 0 \tag{7}$$

The aim of the present study is to define such a relation.

METHOD OF QUADRATURE BY N^{TH} ORDER DIFFERENTIATION

Consider a function f, continuous and with continuous derivatives up to the n^{th} order. The method of quadrature (Lanczos, 1956) consists of approximating the expression

$$A = \int_0^1 f(y)\, dy \tag{8}$$

by

$$A \simeq \frac{1}{C_0^n} \sum_{k=0}^{k=n-1} C^n{}_{k+1} \left[f^{(k)}(0) + (-1)^k f^{(k)}(1)\right] \tag{9}$$

with

$$C_k{}^n = \frac{(2n-k)!}{k!(n-k)!} \tag{10}$$

In Equation (9) $f^{(k)}$ denotes the k^{th} derivative of f.

It was stated by Lanczos (1956) and proved by Solesio (1977) that the quadrature formula (9) always converges. The crux of this method is that A [Equation (9)] depends only on the two end points: $y = 0$ and $y = 1$.

EXPANSIONS OF THE STREAM FUNCTION DISTURBANCE AT THE END POINTS

We will apply the quadrature formula [Equation (8)] to the successive derivative of ϕ. It is then necessary to calculate the values of successive derivatives of ϕ at the two end points $y = 0$ and $y = 1$. Consequently, we will determine the expansions of ϕ for $y = 0$ and $y = 1$.

Assume, at the end point, $y = 0$:

$$\phi_0(y) = \sum_{j=0}^{j=N} a_j y^j \tag{11}$$

By substituting Equation (11) in Equations (1), (4), and (5), we have

$$\sum_{j=4}^{j=N} j(j-1)(j-2)(j-3)\, a_j y^{j-4} - 2\alpha^2 \sum_{j=2}^{j=N} j(j-1)\, a_j y^{j-2} + \alpha^4 \sum_{j=0}^{j=N} a_j y^j$$
$$= i\alpha\, Re \left\{ \left[\frac{3}{2}(1-y^2) - c\right]\left(\sum_{j=2}^{j=N} j(j-1)\, a_j y^{j-2} - \alpha^2 \sum_{j=0}^{j=N} a_j y^j\right) + 3 \sum_{j=0}^{j=N} a_j y^j \right\} \tag{12}$$

$$2\, a_2 + \left(\alpha^2 - \frac{3}{c'}\right) a_0 = 0 \tag{13}$$

$$\frac{\alpha(3 \operatorname{cotg} \theta + \alpha^2 We\, Re)}{c'}\, a_0 + \alpha(Rec' + 3\alpha i)\, a_1 - 6\, i\, a_3 = 0 \tag{14}$$

Equation (12) gives the following recurrence formula for the y^p coefficients:

$$\frac{(p+4)!}{p!} a_{p+4} - 2\,\alpha^2(p+2)(p+1)\, a_{p+2} + \alpha^4 a_p$$
$$= i\alpha\, Re \left\{ \left(\frac{3}{2} - c\right)\left[(p+2)(p+1)\, a_{p+2} - \alpha^2 a_p\right] + 3a_p - \frac{3}{2}\delta_2\left[p(p-1)\, a_p - \alpha^2\, a_{p-2}\right] \right\} \tag{15}$$

with $0 \leqq p \leqq N-4$ and δ_2 defined by

$$\delta_2 = 0 \quad \text{if} \quad p < 2; \quad \delta_2 = 1 \quad \text{if} \quad p \geqq 2$$

Equations (13), (14), and (15) constitute a system of $(N-1)$ homogeneous, linear equations. The unknowns are the $(N+1)$ coefficients a_j $(j = 0, N)$. If we now assume at the end point $y = 1$

$$\phi_1(y) = \sum_{j=0}^{j=N} b_j(y-1)^j \tag{16}$$

the substitution of Equation (16) in Equations (1), (2), and (3) also gives a system of $(N-1)$ homogeneous linear equations with the $(N+1)$ coefficients b_j $(j = 0, N)$ as unknowns. Choosing (a_2, a_3) and (b_2, b_3) as independent variables, we can write $(j = 0, N)$

$$a_j = a_j^2 a_2 + a_j^3 a_3 \tag{17}$$

$$b_j = b_j^2 b_2 + b_j^3 b_3 \tag{18}$$

where a_j^2, a_j^3, b_j^2, b_j^3 $(j = 0, N)$ are functions of the parameters θ, Re, We, α, and c (Solesio, 1977).

Finally, the successive derivatives of the function ϕ at the two end points $y = 0$ and $y = 1$ are given by the following relations $(k = 0, N)$:

$$\phi^{(k)}(0) = \phi_0{}^{(k)}(0) = k! a_k \tag{19}$$

$$\phi^{(k)}(1) = \phi_1{}^{(k)}(1) = k! b_k \tag{20}$$

APPLICATION OF THE QUADRATURE FORMULA

The n^{th} order quadrature formula is then applied to functions ϕ', ϕ'', ϕ''', and ϕ^{iv}. Hence, the values of the first $(n+3)$ derivatives of the function ϕ must be known in $y = 0$ and $y = 1$. Consequently, we will choose $N = n + 3$.

Relation (9) can be written by taking

$$f \triangleq \phi^{(m)} \quad (m = 1, 4)$$

and by using Equations (19) and (20) as follows:

$$(m-1)!(b_{m-1} - a_{m-1}) = \frac{1}{C_o^n} \sum_{k=0}^{k=n-1} C_{k+1}^n \{(m+k)!\left[a_{m+k} + (-1)^k b_{m+k}\right]\} \tag{21}$$

Referring back to Equations (17) and (18), we see that Equation (21) constitutes a homogeneous, linear system of four equations $(m = 1, 4)$ with four unknowns (a_2, a_3, b_2, b_3). This system has a nontrivial solution only if the determinant of the coefficients vanishes. The general element of the matrix of system (21) $d_{L,M}$ can be expressed as follows:

$$d_{L,M} = (L-1)!\Psi_{L-1}^M + \frac{1}{C_o^n} \sum_{k=0}^{k=n-1} C_{k+1}^n (L+k)!\Psi_{L+k}^M \tag{22}$$

with $1 \leqq L \leqq 4$, $1 \leqq M \leqq 4$ in which

$\Psi^1{}_k \triangleq a^2{}_k$; $\Psi^2{}_k \triangleq a^3{}_k$; $\Psi^3{}_{L+k} \triangleq (-1)^k b^2{}_{L+k}$;

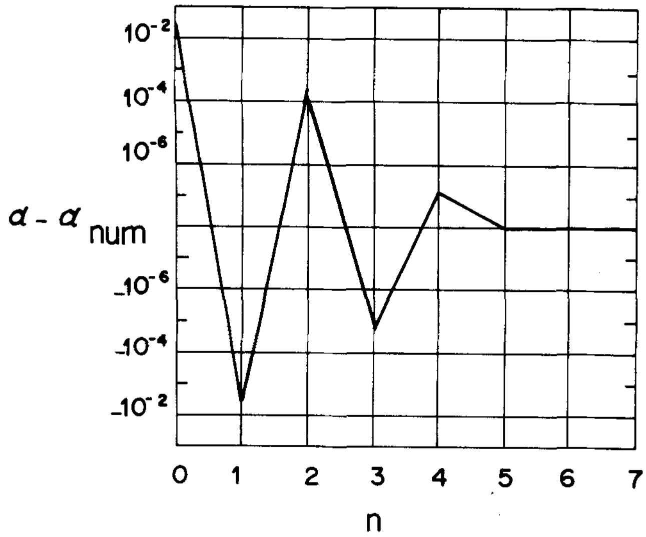

Fig. 1. Value of $(\alpha - \alpha_{num})$ vs. n; $\alpha_{num} = 0.724\,526\,43$ ($Re = 1.29$; $We = 1.9$; $\theta = \pi/2$).

$$\Psi^4{}_{L+k} \triangleq (-1)^k b^3{}_{L+k}; \quad (23)$$

The relation (7) can then be written

$$\mathrm{Det}(d_{L,M}) = 0 \quad (24)$$

RESULTS

A computer program was written to determine, for a given order n and a given set (θ, Re, We), the curves

$$c_r^n = c_r^n(\alpha); \quad \alpha c_i^n = \alpha c_i^n(\alpha)$$

The main advantage of this method is the following. The generalization of the method of quadrature by differentiation to the n^{th} order enables us to obtain the desired precision on the numerical value of the calculated parameters. This possibility is illustrated on the following example. For neutral stability and for a given set (θ, Re, We), the numerical value α_{num} of the wave number α (with a precision of 10^{-8}) is determined. The values of $\alpha - \alpha_{num}$) are plotted in Figure 1 as a function of the order of approximation n.

Sternling and Towell (1965) determined the wave number and the wave velocity corresponding to the neutral stability by means of a numerical integration-iteration technique.

Their results are compared with the predictions of the present method in figures 2 and 3 ($n = 10$). In these two figures, the wave number α and the wave velocity c_r are plotted vs. We^{-1} for a constant value of ζ defined by

$$\zeta \triangleq We Re^{5/3}$$

The wave number data are in very good agreement for wave number values greater than 0.2, whereas a small difference must be noted for the wave number smaller than 0.2. In fact, Yuan, quoted by Yih (1969), showed that the complex velocity is given, for small values of α, by

$$c = 3 + i\alpha\left(\frac{6}{5}Re - \mathrm{cotg}\,\theta\right) + \alpha^2\left(-\frac{12}{7}Re^2 + \frac{10}{7}Re\,\mathrm{cotg}\,\theta - 3\right) + i\alpha^3\left[-\left(\frac{1293}{224} + \frac{We}{3}\right)Re + \frac{9}{5}\mathrm{cotg}\,\theta - \frac{29\,134\,851}{9\,609\,600}Re^3 + \frac{17\,363}{5\,775}Re^2\,\mathrm{cotg}\,\theta - \frac{4}{15}Re\,\mathrm{cotg}^2\,\theta\right] \quad (25)$$

For $We = 1\,000$ we obtain, from Equation (25), $\alpha = 0.059\,48$ and $c_r = 2.989\,35$ from Equation (24), $\alpha = 0.059\,63$ and $c_r = 2.989\,40$; from Figures 2 and 3, $\alpha \simeq 0.10$ and $c_r \simeq 2.96$.

From these results, it appears that the numerical data of Sternling and Towel are not very precise for small values of the wave number. On the contrary, our method seems to give accurate results even for small wave number values.

We have also compared the numerical result of De Bruin (1974) with that predicted by our method. The neutral stability wave number and the correspond-

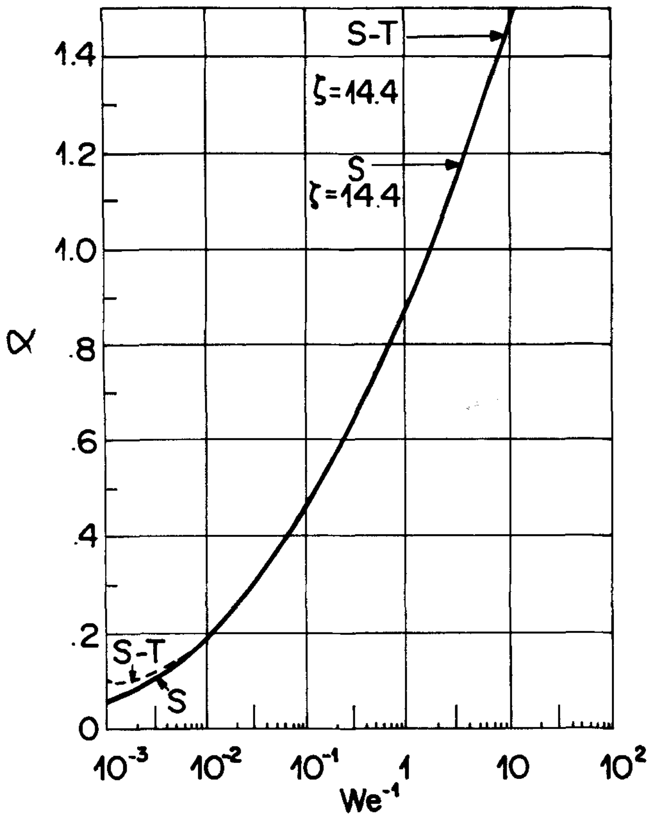

Fig. 2. Neutral stability wave number vs. the reciprocal Weber number ($\zeta = 14.4$; $\theta = \pi/2$). S-T Sternling and Towell; S Solesio.

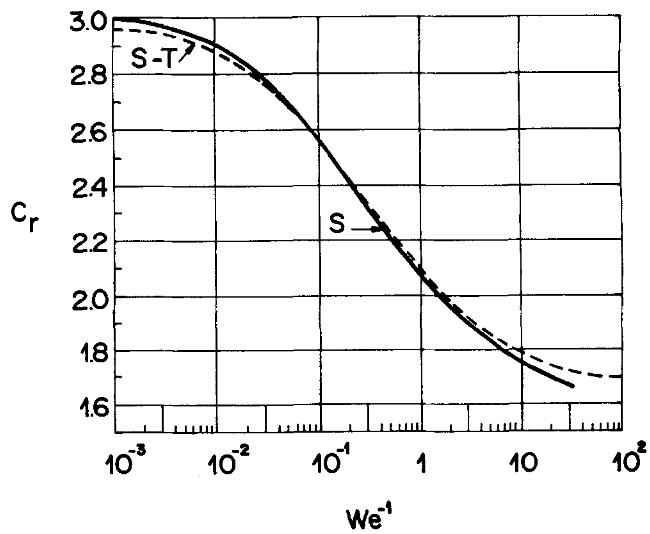

Fig. 3. Neutral stability wave velocity vs. the reciprocal Weber number; S-T Sternling and Towell; S Solesio.

ing wave velocity for $\theta = 1$ deg, $Re = 100$, and $We = 0$ are

De Bruin	$\alpha = 0.359\,7$	$c = 2.721\,15$
Solesio	$\alpha = 0.359\,695\,3$	$c = 2.721\,114\,7$

Nevertheless, for large values of the Reynolds number and for small values of the angle of inclination, the convergence of our method is only secured for small wave number values. For example, for $Re = 5\,000$, $\theta = 1$ deg, and $\zeta = 4\,887$, the convergence is obtained for values of α smaller than 0.15.

CONCLUSIONS

The numerical data obtained by means of the method of quadrature by differentiation are in good agreement with other numerical values and are more accurate for small wave number values than these obtained by Sternling and Towell. Nevertheless, some difficulties still appear with large Re ($\simeq 5\,000$) and small θ ($\theta = 1$ deg).

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NOTATION

c = complex velocity
d = mean film thickness
$d_{L,M}$ = matrix element
n = approximation order of the quadrature formula
Re = Reynolds number $\triangleq u_a{}^* d^* / \nu$
u = axial velocity
We = Weber number $\triangleq \sigma / \rho d^* (\overline{u_a}{}^*)^2$
y = transversal coordinate

Greek Letters

α = wave number
ζ = surface tension group $\triangleq WeRe^{5/3}$
θ = angle of inclination of the plane
ν = kinematic viscosity
ρ = density
σ = surface tension
ϕ = stream function disturbance

Superscripts

$-$ = base flow
$*$ = dimensional quantity
$'$ = differentiation with respect to y
n = approximation order

Subscripts

a = average over the mean thickness of the film

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The Effects of Plasma Constituents on Diffusivity of Oxygen

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There have been several investigations conducted in the past to determine the factors which affect the diffusion of oxygen in blood plasma. The transport of oxygen to and from blood and body tissues is, obviously, necessary for life. Since the oxygen must diffuse through the plasma, it has been thought that the resistance of the plasma layer may be important in determining the rate of oxygen transport, although there is considerable debate over this. For the most part, physiologists have tended to regard diffusion through the plasma as being of minimal importance in the oxygenation and deoxygenation of blood. However, a number of investigators have argued that this may be an important resistance in oxygen transport, perhaps even the controlling one, and more data are needed to determine if this is true or not. This has prompted diffusion studies to determine if oxygen transport is altered by the composition of the plasma.

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